Extended Fourier analysis of signals

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A Fourier transform is a powerful tool of signal analysis and representation of a real or complex-valued function of time $x(t)$ (hereinafter referred to as the signal) in the frequency domain

$$F(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \,, \tag{1.1}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \,, \tag{1.2}$$

where ω is the cyclic frequency. The Fourier transforms orthogonality property

$$\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt = 2\pi\delta(\omega - \omega_0) \tag{2}$$

providing a basis for the signal selective frequency analysis, where $\delta(\omega-\omega_0)$ is the Dirac delta function. Unfortunately, the Fourier transforms calculation according to (1.1) requiring knowledge of the signal $x(t)$ as well as performing of integration operation in infinite time interval. Therefore, for practical evaluation of (1.1) numerically, the signal observation period and the interval of integration is always limited by some finite value Θ, $-\Theta/2 \le t \le \Theta/2$. The same applies to the Fourier analysis of the signal $x(t)$ sampled versions: nonuniformly sampled signal $x(t_k)$ or uniformly sampled signal $x(kT)$, $k= -\infty,\dots,-1,0,1,\dots,+\infty$. Only a finite length sequence $x(t_k)$ or $x(kT)$, $k=0,1,2,\dots,K-1$, are subject of Fourier analysis, where K is a discrete sequence length, T is sampling period and the signal observation period $\Theta=t_{K-1}-t_0$ or $\Theta=KT$. To satisfy the Nyquist limit, uniform sampling of continuous time signal should be performed with the sampling period $T\le\pi/\Omega$, where Ω is upper cyclic frequency of signal $x(t)$. Although nonuniform sampling has no such strict limitation on the mean sampling period $T_s=\Theta/K$, the following analysis we suppose that both sequences, $x(t_k)$ and $x(kT)$, are derived from the band-limited in Ω signal $x(t)$. Let write the basic expressions of the classical and the proposed extended Fourier analysis of continuous time signal $x(t)$ and its sampled versions $x(t_k)$ and $x(kT)$.

Basic expressions of classical Fourier analysis

The classical Fourier analysis dealing with the following finite time Fourier transforms:

$$F_\Theta(\omega) = \int_{-\Theta/2}^{\Theta/2} x(t) e^{-j\omega t} dt \,, \tag{3.1}$$

$$F_\Theta(\omega) = \sum_{k=0}^{K-1} x(t_k) e^{-j\omega t_k} \,, \tag{3.2}$$

$$F_\Theta(\omega) = \sum_{k=0}^{K-1} x(kT) e^{-j\omega kT} \,, \tag{3.3}$$

$$x_\Theta(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F_\Theta(\omega) e^{j\omega t} d\omega \,, \tag{3.4}$$

where (3.4) is the inverse Fourier transform obtained from (1.2) for band-limited in Ω signal.

Transforms (3.2) and (3.3) are known as Discrete Time Fourier Transforms (DTFT) of nonuniformly and uniformly sampled signals. The signal amplitude spectrum is the Fourier transforms (3.1-3.3) results, divided by the observation period Θ,

$$S_\Theta(\omega) = \frac{1}{\Theta} F_\Theta(\omega). \qquad (4)$$

The frequency resolution of the classical Fourier analysis is inversely proportional to the signal observation period Θ.
Obviously, one can get the formula (3.1) by truncation of infinite integration limits in (1.1) and the DTFT (3.2) and (3.3) as result of replacement of infinite sums by finite ones. This mean, the classical Fourier analysis supposed that the signal outside Θ is zeros. In other words, the Fourier transform calculation by formulas (3.1-3.3) is well justified if applied to time-limited within Θ signals. On the other hand, a band-limited in Ω signal cannot be also time-limited and obviously have nonzero values outside Θ. Generally, the Fourier analysis results obtained by using the exponential basis $e^{j\omega t}$, $e^{-j\omega t_k}$ and $e^{-j\omega kT}$ tend to the Fourier transform (1.1), if $\Theta\to\infty$, while in any finite Θ there may exist another transform basis providing a more accurate estimation of (1.1).

Basic expressions of extended Fourier analysis

The idea of extended Fourier analysis is finding the transform basis, applicable for a band-limited signals registered in finite time interval Θ and providing the results as close as possible to the Fourier transform (1.1) defined in infinite time interval. The formulas for proposed extended Fourier analysis could be written as

$$F_\alpha(\omega) = \int_{-\Theta/2}^{\Theta/2} x(t)\alpha(\omega,t)dt, \qquad (5.1)$$

$$F_\alpha(\omega) = \sum_{k=0}^{K-1} x(t_k)\alpha(\omega,t_k), \qquad (5.2)$$

$$F_\alpha(\omega) = \sum_{k=0}^{K-1} x(kT)\alpha(\omega,kT), \qquad (5.3)$$

$$x_\alpha(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F_\alpha(\omega)e^{j\omega t}d\omega, \qquad (5.4)$$

where in general case the transform basis $\alpha(\omega,t)$, $\alpha(\omega,t_k)$ and $\alpha(\omega,kT)$ are not equal to the classical ones (3.1-3.3). Note that the inverse Fourier transform (5.4) still holds the exponential basis $e^{j\omega t}$. To ensure that the results of transforms (5.1-5.3) are close to the result of the Fourier transform (1.1) for the signal $x(t)$, the following minimum least squares expression will be composed and solved

$$\left|F(\omega) - F_\alpha(\omega)\right|^2 \to \min. \qquad (6)$$

Unfortunately, as already stated above, the calculation of $F(\omega)$ for a band-limited signal cannot be performed directly. So, in order to compose (6), we should find an adequate substitution. Let's recall that a complex exponent (known as an analytic signal), at cyclic frequency ω_0 and with a complex amplitude $S(\omega_0)$, is defined in infinite time interval as

$$x(\omega_0,t) = S(\omega_0)e^{j\omega_0 t}, -\infty < t < \infty. \qquad (7)$$

The Fourier transform of an analytic signal can be expressed by the Dirac delta function (2)

$$\int_{-\infty}^{\infty} x(\omega_0, t)\, e^{-j\omega t}\, dt = 2\pi S(\omega_0)\delta(\omega - \omega_0)\,. \tag{8}$$

Now, let's use (7) as a signal model with known amplitude spectrum $S(\omega_0)$ for frequencies in range $-\Omega \le \omega_0 \le \Omega$ and, in the minimum least square expression (6), substitute $F(\omega)$ by the signal model Fourier transform (8) and the signals $x(t)$, $x(t_k)$ and $x(kT)$ in (5.1-5.3) by the signal models (7), correspondingly. Finally, the minimum least square error estimators for all the three signal cases get the following form

$$\Delta = \int_{-\Omega}^{\Omega} \left| 2\pi S(\omega_0)\delta(\omega - \omega_0) - \int_{-\Theta/2}^{\Theta/2} S(\omega_0)\, e^{j\omega_0 t}\, \alpha(\omega, t) dt \right|^2 d\omega_0\,, \tag{9.1}$$

$$\Delta = \int_{-\Omega}^{\Omega} \left| 2\pi S(\omega_0)\delta(\omega - \omega_0) - \sum_{k=0}^{K-1} S(\omega_0) e^{j\omega_0 t_k} \alpha(\omega, t_k) \right|^2 d\omega_0\,, \tag{9.2}$$

$$\Delta = \int_{-\Omega}^{\Omega} \left| 2\pi S(\omega_0)\delta(\omega - \omega_0) - \sum_{k=0}^{K-1} S(\omega_0) e^{j\omega_0 kT} \alpha(\omega, kT) \right|^2 d\omega_0\,. \tag{9.3}$$

The solutions of (9.1-9.3) for a definite signal model (7) provide the basis $\alpha(\omega,t)$, $\alpha(\omega,t_k)$ and $\alpha(\omega,kT)$ for extended Fourier transforms (5.1-5.3). To control how close the selected signal model amplitudes $S(\omega_0)$ are to the signals $x(t)$, $x(t_k)$ and $x(kT)$ amplitude spectrum, we will find the formulas for estimate signal amplitude spectrum $S_\alpha(\omega)$ in the extended Fourier basis $\alpha(\omega,t)$, $\alpha(\omega,t_k)$ and $\alpha(\omega,kT)$.

The formula (8) is showing the connection between the signal model Fourier transform and its amplitude spectrum, from where $S(\omega_0)$ could be expressed as signal model Fourier transform divided by $2\pi\delta(\omega - \omega_0)$. Taking (8) into account, $S_\alpha(\omega)$ is calculated as the transforms (5.1-5.3) divided by the estimate of $2\pi\delta(\omega - \omega_0)$ in the extended Fourier basis, which is determined from (9.1-9.3) in the case Δ=0 and $\omega_0=\omega$,

$$S_\alpha(\omega) = \frac{\int_{-\Theta/2}^{\Theta/2} x(t)\alpha(\omega, t) dt}{\int_{-\Theta/2}^{\Theta/2} e^{j\omega t}\alpha(\omega, t) dt}\,, \tag{10.1}$$

$$S_\alpha(\omega) = \frac{\sum_{k=0}^{K-1} x(t_k)\alpha(\omega, t_k)}{\sum_{k=0}^{K-1} e^{j\omega t_k}\alpha(\omega, t_k)}\,, \tag{10.2}$$

$$S_\alpha(\omega) = \frac{\sum_{k=0}^{K-1} x(kT)\alpha(\omega, kT)}{\sum_{k=0}^{K-1} e^{j\omega kT}\alpha(\omega, kT)}\,. \tag{10.3}$$

Values of the denominator in formulas (10.1-10.3) are in inverse ratio to the frequency resolution of the extended Fourier transform. For example, after substituting exponential basis $\alpha(\omega, t) = e^{-j\omega t}$ in (10.1), the denominator becomes equal to Θ as in formula (4) for the classical Fourier analysis. To establish relationships between classical and extended Fourier analysis, let's consider a special case of Δ estimators (9.1-9.3) for the signal model having a rectangular form of amplitude spectrum, $S(\omega_0)$=1 for $-\Omega \le \omega_0 \le \Omega$ and zeros outside.

Extended Fourier analysis: a particular solution

The minimum least square error estimators (9.1-9.3) for the signal model $S(\omega_0)=1$, $-\Omega\le\omega_0\le\Omega$ and zeros outside, reduces to

$$\Delta = \int_{-\Omega}^{\Omega} \left| 2\pi\delta(\omega-\omega_0) - \int_{-\Theta/2}^{\Theta/2} e^{j\omega_0 t}\alpha(\omega,t)dt \right|^2 d\omega_0 , \tag{11.1}$$

$$\Delta = \int_{-\Omega}^{\Omega} \left| 2\pi\delta(\omega-\omega_0) - \sum_{k=0}^{K-1} e^{j\omega_0 t_k}\alpha(\omega,t_k) \right|^2 d\omega_0 , \tag{11.2}$$

$$\Delta = \int_{-\Omega}^{\Omega} \left| 2\pi\delta(\omega-\omega_0) - \sum_{k=0}^{K-1} e^{j\omega_0 kT}\alpha(\omega,kT) \right|^2 d\omega_0 . \tag{11.3}$$

The solution of (11.1) for continuous time signal $x(t)$ is found as a partial derivation $\frac{\partial\Delta}{\partial\alpha(\omega,\tau)} = 0,\ -\Theta/2 \le \tau \le \Theta/2$, and leads to the linear integral equation

$$\int_{-\Theta/2}^{\Theta/2} \frac{\sin(\Omega(t-\tau))}{\pi(t-\tau)}\alpha(\omega,t)dt = e^{-j\omega\tau} . \tag{12}$$

Step by step solution of (12) is given in **[1]** and **[5]**. Finally, the basis $\alpha(\omega,t)$ are found by applying a specific functions system - a prolate spheroidal wave functions $\psi_k(t)$, $k=0,1,2,...$ and are written as series expansion

$$\alpha(\omega,t) = \sum_{k=0}^{\infty} \frac{B_k(\omega)}{\lambda_k}\psi_k(t) . \tag{13}$$

The extended Fourier Transform of continuous time signal $x(t)$ are given by

$$F_\alpha(\omega) = \sum_{k=0}^{\infty} B_k(\omega)a_k ,\ -\Omega \le \omega \le \Omega , \tag{14.1}$$

$$x_\alpha(t) = \sum_{k=0}^{\infty} \psi_k(t)a_k ,\ -\infty < t < \infty , \tag{14.2}$$

$$S_\alpha(\omega) = \frac{\sum_{k=0}^{\infty} B_k(\omega)a_k}{\sum_{k=0}^{\infty} \left|B_k(\omega)\right|^2} , \tag{14.3}$$

where $a_k = \frac{1}{\lambda_k}\int_{-\Theta/2}^{\Theta/2} x(\tau)\psi_k(\tau)d\tau$, $\lambda_k = \int_{-\Theta/2}^{\Theta/2} \psi_k^2(t)dt$, $B_k(\omega) = \sqrt{\frac{\pi\Theta}{\lambda_k\Omega}}\psi_k\left(\omega\frac{\Theta}{2\Omega}\right)(-j)^k$.

The extended Fourier transform in accordance with (14.1) requesting a calculations of infinite sums, this mean, an infinite quantity of mathematical operations, therefore it's impossible for real world applications. Theoretically, the value of denominator $\sum_{k=0}^{K}\left|B_k(\omega)\right|^2$ in amplitude spectrum formula (14.3) tends to infinite for $K\to\infty$, and the extended Fourier transform (14.1) provide a supper-resolution - an ability to determine the Fourier transform for the sum of sinusoids or complex exponents, if frequencies of them differ by arbitrary small finite value.

The detailed solution steps for the minimum least square error estimators (11.2) and (11.3) are given in articles **[2]** and **[3]**. Similarly to (11.1), finding of the partial derivations $\frac{\partial\Delta}{\partial\alpha(\omega,t_l)} = 0$ and $\frac{\partial\Delta}{\partial\alpha(\omega,lT)} = 0,$ for $l = 0,1,2,...,K-1$, leads to the system of linear equations

$$\sum_{k=0}^{K-1} \frac{\sin(\Omega(t_k - t_l))}{\pi(t_k - t_l)} \alpha(\omega, t_k) = e^{-j\omega t_l}, \tag{15.1}$$

$$\sum_{k=0}^{K-1} \frac{\sin(\Omega T(k-l))}{\pi T(k-l)} \alpha(\omega, kT) = e^{-j\omega lT}. \tag{15.2}$$

The solution of (15) expressed in matrix form is

$$\mathbf{A}_\omega = \mathbf{R}^{-1}\mathbf{E}_\omega, \tag{16}$$

where $\mathbf{A}_\omega$ (Kx1) and $\mathbf{E}_\omega$ (Kx1) are the extended Fourier and the exponential basis. The formulas of the Extended Discrete Time Fourier Transform (EDTFT) for signal model $S(\omega_0)$=1, $-\Omega \le \omega_0 \le \Omega$, are derived by substituting of transform basis (16) into expressions (5) and (10)

$$F_\alpha(\omega) = \mathbf{x}\mathbf{R}^{-1}\mathbf{E}_\omega, \quad -\Omega \le \omega \le \Omega, \tag{17.1}$$

$$x_\alpha(t) = \mathbf{x}\mathbf{R}^{-1}\mathbf{E}_t, \quad -\infty < t < \infty, \tag{17.2}$$

$$S_\alpha(\omega) = \frac{\mathbf{x}\mathbf{R}^{-1}\mathbf{E}_\omega}{\mathbf{E}_\omega^H \mathbf{R}^{-1}\mathbf{E}_\omega}, \tag{17.3}$$

where $(.)^{-1}$ and $(.)^H$ denotes the inverse and the Hermitian (complex conjugate) transpose. The matrices for nonuniformly sampled signal case are composed as follows

$\mathbf{x}$ (1xK) - $x(t_k)$, $\mathbf{E}_\omega$ (Kx1) - $e^{-j\omega t_l}$, $\mathbf{R}$ (KxK) - $r_{l,k} = \dfrac{\sin\Omega(t_k - t_l)}{\pi(t_k - t_l)}$ and $\mathbf{E}_t$ (Kx1) - $\dfrac{\sin\Omega(t - t_l)}{\pi(t - t_l)}$.

Uniformly sampled signal $x(kT)$ can be considered as a special case of nonuniform sampling at time moments $t_k=kT$, k=0,1,2,…,K-1. Then the matrices elements in (16, 17) are

$\mathbf{x}$ (1xK) - $x(kT)$, $\mathbf{E}_\omega$ (Kx1) - $e^{-j\omega lT}$, $\mathbf{R}$ (KxK) - $r_{l,k} = \dfrac{\sin\Omega T(k-l)}{\pi T(k-l)}$, $\mathbf{E}_t$ (Kx1) - $\dfrac{\sin\Omega(t - lT)}{\pi(t - lT)}$.

In particular, if sampling of signal $x(kT)$ is done with Nyquist rate, $T=\pi/\Omega$, then the matrix **R** becomes a unit matrix **I** and the formula (17.1) coincide with classical DTFT (3.3), but the formula (17.3) reduces to well known relationship between discrete signal Fourier transform and its amplitude spectrum

$$S_a(\omega) = \frac{1}{K} F_a(\omega). \tag{18}$$

In case, mean sampling period is less then it demands by Nyquist criteria for uniformly sampled signal, $T<\pi/\Omega$, the EDTFT approach can provide a high frequency resolution and improved spectral estimation quality. Unfortunate an achievement of such results is limited by finite precision in the mathematical calculations and by restrictions on frequency range in the process of signal sampling. Theoretical value of denominator in (17.3) $\mathbf{E}_\omega^H \mathbf{R}^{-1}\mathbf{E}_\omega = K$ and the frequency resolution should increase proportionally to the number of samples in the signal observation period Θ. In the border-case, if number of samples within Θ increasing infinitely, $K\to\infty$, and the discrete time signal tends to the continuous time signal $x(t)$, the EDTFT (17.1) gives the same results as (14.1).

Extended DTFT

Now, let consider the solution of the minimum least square error estimators (9.2) and (9.3) for arbitrary selected signal model $S(\omega_0)$ (see also **[2]**, **[3]**). The derivation formulas for both estimators are similar to ones given in previous section. For example, a partial derivation of (9.2) by basis functions $\dfrac{\partial\Delta}{\partial\alpha(\omega, t_l)} = 0$, for $l = 0,1,2,...,K-1$ provide the least square solution

$$\int_{-\Omega}^{\Omega}\left(2\pi S(\omega_0)\delta(\omega-\omega_0)-\sum_{k=0}^{K-1}S(\omega_0)e^{j\omega_0 t_k}\alpha(\omega,t_k)\right)S^*(\omega_0)e^{-j\omega_0 t_l}d\omega_0=0\,, \qquad (19)$$

where (.)* denote the complex conjugate value. Equation (19) can be rewritten as

$$\sum_{k=0}^{K-1}\left(\int_{-\Omega}^{\Omega}\left|S(\omega_0)\right|^2 e^{j\omega_0(t_k-t_l)}d\omega_0\right)\alpha(\omega,t_k)=2\pi\int_{-\Omega}^{\Omega}\left|S(\omega_0)\right|^2 e^{-j\omega_0 t_l}\delta(\omega-\omega_0)d\omega_0\,. \qquad (20)$$

The filtering feature of Dirac delta function $\int_{-\infty}^{\infty}f(x)\delta(x-x_0)dx=f(x_0)$ applied to the right part of (20) gives the final form of the system of linear equations

$$\sum_{k=0}^{K-1}\left(\frac{1}{2\pi}\int_{-\Omega}^{\Omega}\left|S(\omega_0)\right|^2 e^{j\omega_0(t_k-t_l)}d\omega_0\right)\alpha(\omega,t_k)=\left|S(\omega)\right|^2 e^{-j\omega t_l}\,, \qquad (21.1)$$

$$\sum_{k=0}^{K-1}\left(\frac{1}{2\pi}\int_{-\Omega}^{\Omega}\left|S(\omega_0)\right|^2 e^{j\omega_0 T(k-l)}d\omega_0\right)\alpha(\omega,kT)=\left|S(\omega)\right|^2 e^{-j\omega lT}\,, \qquad (21.2)$$

for l=0,1,2,...,K-1, where $\left|S(\omega)\right|^2$ is the signal model power at $\omega_0=\omega$. The equations (21.2) are applicable for uniformly sampled signal $x(kT)$ and can be derived from (9.3) in a similar way as (21.1). The EDTFT basis $\mathbf{A}_\omega$ (Kx1) - $\alpha(\omega,t_k)$ or $\alpha(\omega,kT)$ are found as a solution of (21)

$$\mathbf{A}_\omega=\left|S(\omega)\right|^2\mathbf{R}^{-1}\mathbf{E}_\omega\,. \qquad (22)$$

Substituting of transform basis (22) into expressions (5) and (10), yields the formulas for calculation of the EDTFT:

$$F_\alpha(\omega)=\mathbf{x}\mathbf{A}_\omega=\left|S(\omega)\right|^2\mathbf{x}\mathbf{R}^{-1}\mathbf{E}_\omega,\quad -\Omega\le\omega\le\Omega, \qquad (23.1)$$

$$x_\alpha(t)=\mathbf{x}\mathbf{R}^{-1}\mathbf{E}_t,\quad -\infty<t<\infty, \qquad (23.2)$$

$$S_\alpha(\omega)=\frac{\mathbf{x}\mathbf{A}_\omega}{\mathbf{E}_\omega^H\mathbf{A}_\omega}=\frac{\mathbf{x}\left(\left|S(\omega)\right|^2\mathbf{R}^{-1}\mathbf{E}_\omega\right)}{\mathbf{E}_\omega^H\left(\left|S(\omega)\right|^2\mathbf{R}^{-1}\mathbf{E}_\omega\right)}=\frac{\mathbf{x}\mathbf{R}^{-1}\mathbf{E}_\omega}{\mathbf{E}_\omega^H\mathbf{R}^{-1}\mathbf{E}_\omega}\,. \qquad (23.3)$$

The elements of matrices **R** and $\mathbf{E}_t$ in the formulas (22, 23.1-23.3) are expressed by integrals

R (KxK) **-** $r_{l,k}=\frac{1}{2\pi}\int_{-\Omega}^{\Omega}\left|S(\omega_0)\right|^2 e^{j\omega_0(t_k-t_l)}d\omega_0$, $\mathbf{E}_t$ (Kx1) - $\frac{1}{2\pi}\int_{-\Omega}^{\Omega}\left|S(\omega)\right|^2 e^{j\omega(t-t_l)}d\omega$,

for nonuniformly sampled signal **x** (1xK) - $x(t_k)$, and

R (KxK) **-** $r_{l,k}=\frac{1}{2\pi}\int_{-\Omega}^{\Omega}\left|S(\omega_0)\right|^2 e^{j\omega_0(k-l)T}d\omega_0$, $\mathbf{E}_t$ (Kx1) - $\frac{1}{2\pi}\int_{-\Omega}^{\Omega}\left|S(\omega)\right|^2 e^{j\omega(t-lT)}d\omega$,

for case of uniformly sampled signal **x** (1xK) - $x(kT)$, where k,l=0,1,2,…,K-1. In contrary to (17.1), $F_\alpha(\omega)$ estimate (23.1) depends on signal power spectrum at the frequency of analysis, ω, while amplitude spectrum $S_\alpha(\omega)$ (23.3) do not depend on value of $\left|S(\omega)\right|^2$ which is canceled out in the numerator and denominator.

The frequency resolution of the EDTFT is in inverse ration to $\left|S(\omega)\right|^2\mathbf{E}_\omega^H\mathbf{R}^{-1}\mathbf{E}_\omega$ and varied in the frequency range $-\Omega\le\omega\le\Omega$.

Calculation of the EDTFT by formulas (23) requires knowledge of the signal model spectrum which generally is not known. At the same time, the amplitude spectrum obtained in the previous section by the formula (17.3) can be used as a source of such information. This suggests the following iterative algorithm, where the signal model amplitude spectrum $S(\omega_0)$ tends to the signal amplitude spectrum $S_\alpha(\omega)$:

Iteration 1. Calculate $S_a^{(1)}(\omega)$ (17.3) applying default signal model $S(\omega_0)$=1.
Iteration 2. Calculate $S_a^{(2)}(\omega)$ (23.3) by using the signal model $S_a^{(1)}(\omega_0)$.
Iteration 3. Calculate $S_a^{(3)}(\omega)$ (23.3) by using the signal model $S_a^{(2)}(\omega_0)$.
...
Iteration i. Calculate $S_a^{(i)}(\omega)$ (23.3) by using the signal model $S_a^{(i-1)}(\omega_0)$.
Iterations are repeated until $S_a^{(i)}(\omega) \approx S_a^{(i-1)}(\omega)$.
The EDTFT output $F_a^{(I)}(\omega)$ (23.1) is calculated for the last performed iteration *I*.
By default the signal model $S(\omega_0)$=1 is used as input of the EDTFT algorithm. However, additional information about the signal to be analyzed can be used to create a more realistic signal model for the EDTFT input and to reduce the number of iterations required to reach the stopping iteration criteria.

Extended DFT algorithm

The EDTFT considered in previous sections is a function of continuous frequency ($-\Omega \le \omega \le \Omega$), while described below the Extended DFT (EDFT) algorithm calculated the EDTFT on a discrete frequency set $-\Omega \le 2\pi f_n < \Omega$, n=0,1,2,…,N-1. The number of frequency points $N \ge K$ should be selected sufficiently great to substitute the integrals used for calculation of matrix **R** (KxK) in the expressions (22, 23) by the finite sums:

$$r_{l,k} = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} |S(\omega_0)|^2 e^{j\omega_0(t_k - t_l)} d\omega_0 \approx \frac{1}{N}\sum_{n=0}^{N-1} |S(f_n)|^2 e^{j2\pi f_n (t_k - t_l)}, \tag{24.1}$$

$$r_{l,k} = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} |S(\omega_0)|^2 e^{j\omega_0(k-l)T} d\omega_0 \approx \frac{1}{N}\sum_{n=0}^{N-1} |S(f_n)|^2 e^{j2\pi f_n (k-l)T}, \tag{24.2}$$

where l,k=0,1,2,…,K-1 and K is the length of nonuniformly or uniformly sampled sequence **x**. The matrix (24.1)

$$\mathbf{R} = \begin{bmatrix} r_{0,0}(0) & r_{0,1}(t_1 - t_0) & r_{0,2}(t_2 - t_0) & \dots & r_{0,K-1}(t_{K-1} - t_0) \\ r_{1,0}(t_0 - t_1) & r_{1,1}(0) & r_{1,2}(t_2 - t_1) & \dots & r_{1,K-1}(t_{K-1} - t_1) \\ r_{2,0}(t_0 - t_2) & r_{2,1}(t_1 - t_2) & r_{2,2}(0) & \dots & r_{2,K-1}(t_{K-1} - t_2) \\ \dots & \dots & \dots & \dots & \dots \\ r_{K-1,0}(t_0 - t_{K-1}) & r_{K-1,1}(t_1 - t_{K-1}) & r_{K-1,2}(t_2 - t_{K-1}) & \dots & r_{K-1,K-1}(0) \end{bmatrix}, \tag{25}$$

for nonuniformly sampled signal case possesses Hermitian symmetry, $r_{l,k} = r_{k,l}^*$, but (24.2) for uniformly sampled signal is a Hermitian Toepltiz matrix

$$\mathbf{R} = \begin{bmatrix} r_{0,0}(0) & r_{0,1}(T) & r_{0,2}(2T) & \dots & r_{0,K-1}((K-1)T) \\ r_{1,0}(-T) & r_{1,1}(0) & r_{1,2}(T) & \dots & r_{1,K-1}((K-2)T) \\ r_{2,0}(-2T) & r_{2,1}(-T) & r_{2,2}(0) & \dots & r_{2,K-1}((K-3)T) \\ \dots & \dots & \dots & \dots & \dots \\ r_{K-1,0}(-(K-1)T) & r_{K-1,1}(-(K-2)T) & r_{K-1,2}(-(K-3)T) & \dots & r_{K-1,K-1}(0) \end{bmatrix}, \tag{26}$$

where $r_{l,k}$ representing the autocorrelation function of the selected signal model and can be calculated by applying Inverse DFT (IDFT) to the signal model power spectrum $|S(f_n)|^2$. In the case where the signal and its model power spectra are close, $|S_\alpha(f_n)|^2 \approx |S(f_n)|^2$, (24) is also an estimate of the autocorrelation function for the sequence **x**.
The EDFT can be expressed by the following iterative algorithm

$$\mathbf{R} = \frac{1}{N}\mathbf{E}\mathbf{W}^{(i)}\mathbf{E}^{H}, \tag{27.1}$$

$$\mathbf{F}^{(i)} = \mathbf{x}\mathbf{A}^{(i)} = \mathbf{x}\mathbf{R}^{-1}\mathbf{E}\mathbf{W}^{(i)}, \tag{27.2}$$

$$\mathbf{S}^{(i)} = \frac{\mathbf{x}\mathbf{R}^{-1}\mathbf{E}}{diag(\mathbf{E}^{H}\mathbf{R}^{-1}\mathbf{E})}, \tag{27.3}$$

$$\mathbf{W}^{(i+1)} = diag(|\mathbf{S}^{(i)}|^{2}) \tag{27.4}$$

for the iteration number i=1,2,3,…I, where the matrix $\mathbf{E}$ (KxN) has elements $e^{-j2\pi f_n t_k}$ and $diag(\mathbf{E}^{H}\mathbf{R}^{-1}\mathbf{E})$ (1xN) means extracting the main diagonal elements from quadratic matrix. By default the diagonal weight matrix $\mathbf{W}^{(i)}$ (NxN) for the first iteration is a unit matrix, $\mathbf{W}^{(1)} = \mathbf{I}$. If other diagonal matrix is used then it must have at least K nonzero elements. The IDFT

$$\mathbf{x} = \frac{1}{N}\mathbf{F}^{(i)}\mathbf{E}^{H} \tag{28}$$

can be applied to output $\mathbf{F}^{(i)}$ and return back original K-samples of uniform or nonuniform sequence $\mathbf{x}$. Since the length of the frequency set $N \geq K$, then (28) can be modified to obtain a sequence $\mathbf{x}_{\alpha}$ (1xN) - $x_a(t_m)$, m=0,1,2,…,N-1,

$$\mathbf{x}_{\alpha} = \frac{1}{N}\mathbf{F}^{(i)}\mathbf{E}_{N}^{H}, \tag{29}$$

where exponents matrix $\mathbf{E}_N$ (NxN) has elements $e^{-j2\pi f_n t_m}$. The reconstructed by the formula (29) sequence is the original sequence plus forward and backward extrapolation of $\mathbf{x}$ to length N and/or interpolation if there are gaps inside of $\mathbf{x}$. The maximum frequency resolution of the iterative algorithm is limited by the length N of frequency set, not by the length K of sequence as in application of classical DFT. This mean, the EDFT is able to increase the frequency resolution N/K times in comparison with classical DFT. This can be verified by comparing the diagonal elements of the product of IDFT and DFT basis, $diag(\frac{1}{N}\mathbf{E}^{H}\mathbf{E}) = K/N < 1$, with the relationship, $0 < diag(\frac{1}{N}\mathbf{E}^{H}\mathbf{A}^{(i)}) = \frac{1}{N}\mathbf{F}^{(i)}/\mathbf{S}^{(i)} \leq 1$, corresponding to the IDFT and EDFT basis $\mathbf{A}^{(i)}$ from (27.2). At the same time there is a restriction on the frequency resolution $sum(\mathbf{F}^{(i)}/\mathbf{S}^{(i)}) = NK$, which is satisfied by iteration, and in order to achieve a high resolution at certain frequencies, the EDFT must decrease the resolution on other frequencies. In a border-case N=K, the iterative algorithm output do not depend on weight matrix $\mathbf{W}$ and the optimal EDFT basis can be found in a non-iterative way (as result of the first EDFT iteration).

Computer simulations

The computer modeling results are presented for the complex-value test signal consisting of three non-overlapping components symbolized in Figure 1a and 1b in red color. The uniform and nonuniform sequences of length 64 samples are derived by simulating 12-bit ADC (analog-to-digital converter) of composite test signal. True spectrum of composite test signal consisting of a band-limited noise in frequency range [-0.5...-0.25], the rectangular impulse in range [0...0.25] and complex exponent at frequency 0.35. The third test sequence is well-known Marple&Kay data set - 64-point real sample sequence from a process consisting of three sinusoids at frequencies 0.1, 0.2 and 0.21 (magnitudes 0.1, 1.0 and 1.0) and a colored noise in frequency range [0.2…0.5] (see red color plot in Figure 1c). More details could be found in S.M. Kay and S.L. Marple article "Spectrum analysis - a modern perspective", Proc.IEEE, No.11, Vol.69, 1981. Figure 1 displays the results of the DFT (in blue color) and the 10th iteration of the Extended DFT (in black color). The number of frequencies (length of the DFT) is chosen to be equal 1000, which gives spectral estimates with normalized frequency step 0.001.

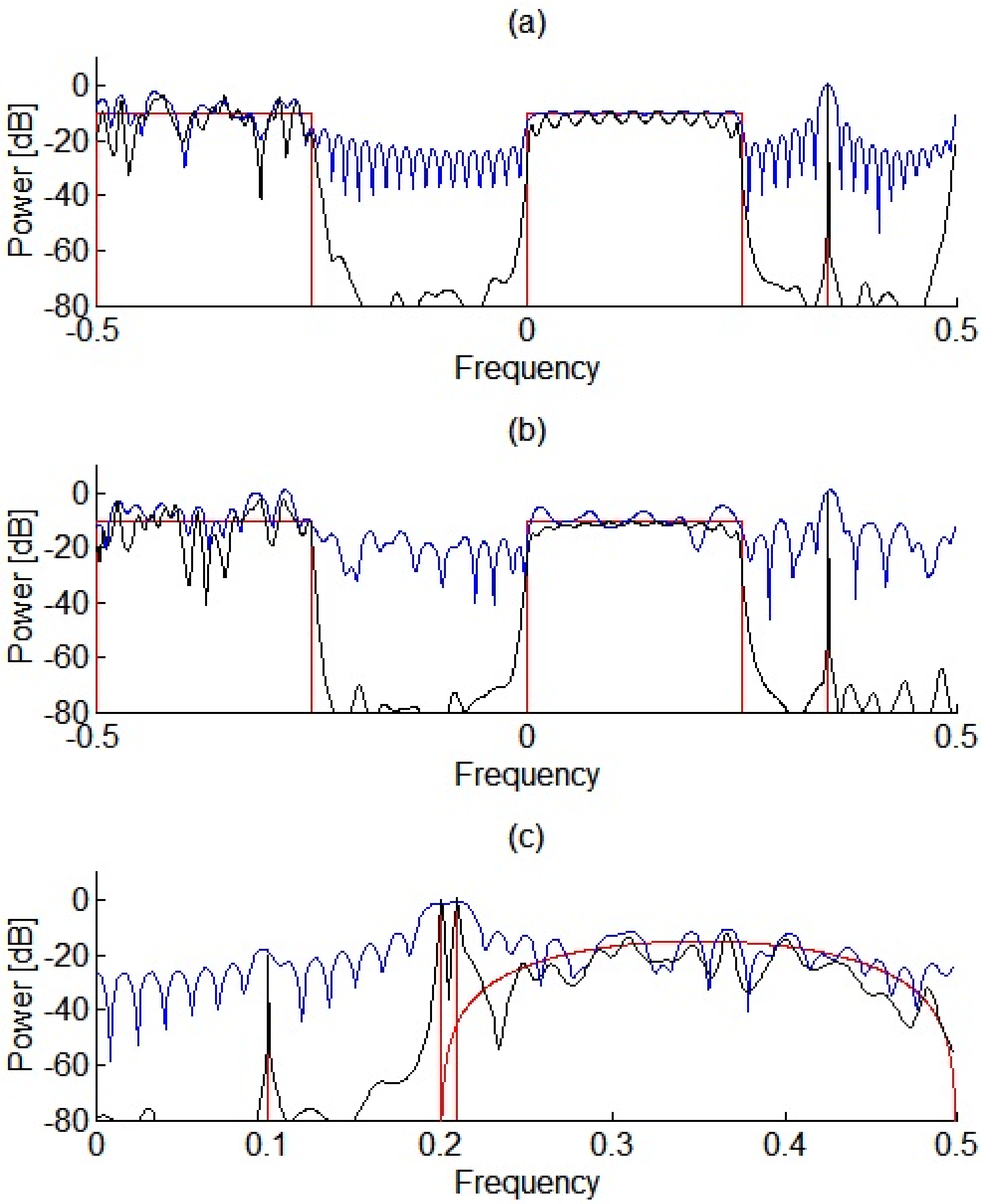

Figure 1. DFT (blue) and Extended DFT (10th iteration) Power Spectrum estimate of
(a) - uniform complex-value test sequence,
(b) - nonuniform complex-value test sequence,
(c) - Marple&Kay real-value sequence.
(Launch attached MATLAB program EDFT_FIG.m to recreate the simulations.)

The EDFT output illustrate, that the proposed algorithm providing a high-frequency resolution, is able to estimate a composite signal spectrum, and is working equally well for uniformly and nonuniformly sampled signal. Blue color plots in Figure 1c showing that due to limited frequency resolution the DFT cannot resolve signal components at frequencies 0.2 and 0.21. Although the first EDFT iteration coincides with the DFT, in subsequent iterations EDFT is able to increase the frequency resolution and all three sinusoids are clearly distinguished in the 10th iteration results. Comprehensive computer simulation results of the proposed EDFT algorithm and comparison with other spectral analysis methods for test signals are given in **[2]** and **[5]**.

Extended DFT and other nonparametric approaches

In the previous sections, starting with the Fourier integral (1) and using its orthogonality property (2), by establishing and solving the minimum least square error estimators (9), the Extended DFT algorithm is obtained analytically, and computer simulation results conforming to the capabilities of the original algorithm are given. Let's compare the EDFT with known nonparametric methods - Capon filter, Generalized Least Squares (GLS) solution and High-Resolution DFT introduced by Sacchi, Ulrych and Walker in 1998, and try to prove or disprove the possibility of derivation of an iterative EDFT algorithm based on these approaches.

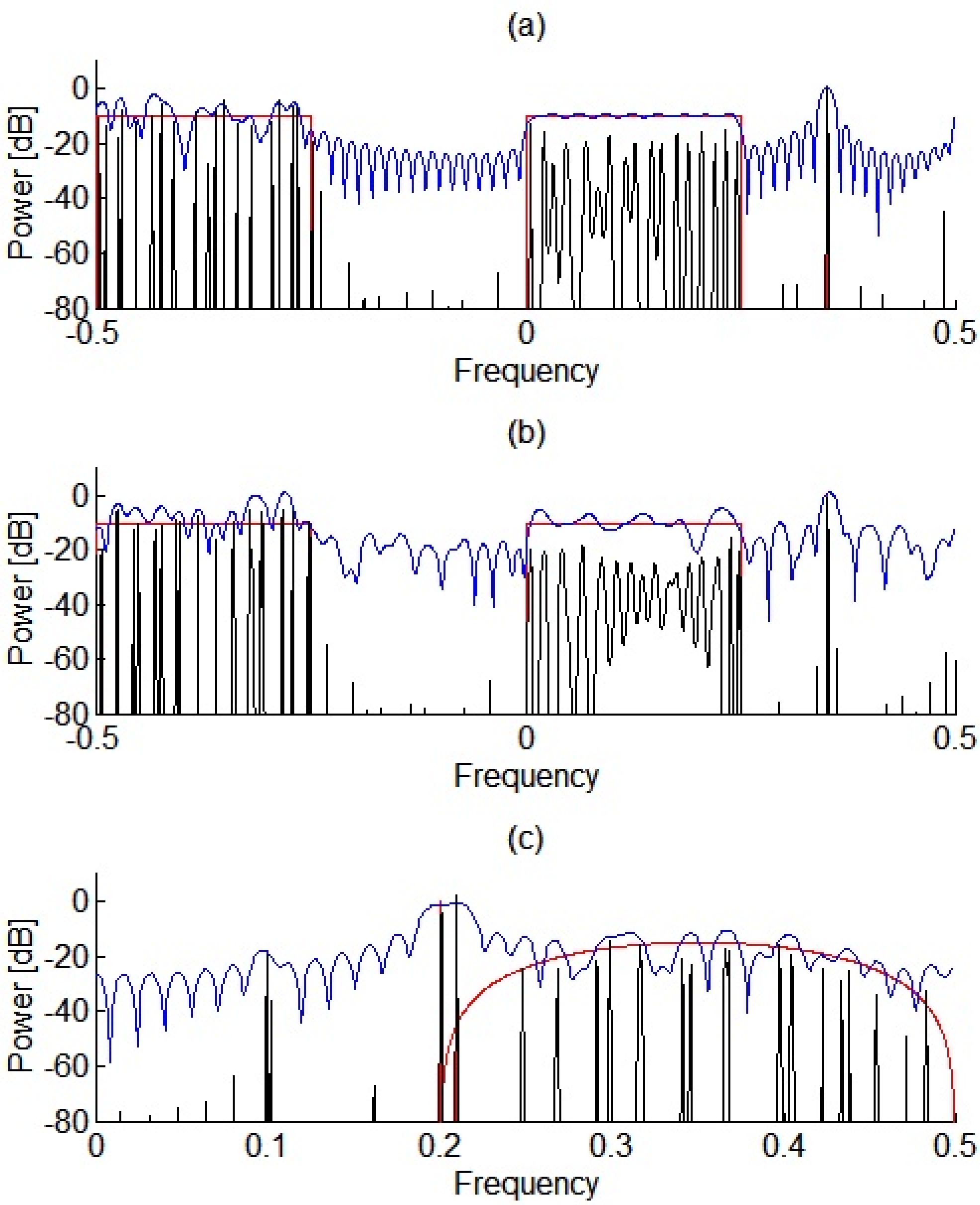

Figure 2. DFT (blue) and High-Resolution DFT (10th iteration) Power Spectrum estimate of
(a) - uniform complex-value test sequence,
(b) - nonuniform complex-value test sequence,
(c) - Marple&Kay real-value sequence.

The Capon filter also known as Minimum Variance spectrum estimate can be viewed as the output of a bank of filters with each filter centered at one of the analysis frequencies

$$y_{\omega}(nT) = \sum_{k=0}^{K-1} x\big((n-k)T\big) h_{\omega}(kT) = \tilde{\mathbf{x}}\mathbf{h}_{\omega}\,. \tag{30}$$

In the matrix notation $\tilde{\mathbf{x}} = \big[x(nT), x\big((n-1)T\big), x\big((n-2)T\big),..., x\big((n-K)T\big)\big]$ is the filter input signal and $\mathbf{h}_{\omega} = \big[h_{\omega}(0), h_{\omega}(T), h_{\omega}(2T),..., h_{\omega}\big((K-1)T\big)\big]^T$ is the filter coefficients. Here, and in the following, superscripts $(.)^T, (.)^*, (.)^H$ denote transpose, complex conjugate, Hermitian transpose of the vector or matrix and the subscript ω is used to indicate a dependence on the filter's center frequency.

The Capon filter is designed to minimize the variance on the filter output

$$\sigma_y^2 = \mathcal{E}\left\{\left|y_{\omega}(nT)\right|^2\right\} = \mathcal{E}\left\{y_{\omega}^H(nT)\, y_{\omega}(nT)\right\} = \mathcal{E}\left\{\mathbf{h}_{\omega}^H \tilde{\mathbf{x}}^H \tilde{\mathbf{x}}\mathbf{h}_{\omega}\right\} = \mathbf{h}_{\omega}^H \mathcal{E}\left\{\tilde{\mathbf{x}}^H \tilde{\mathbf{x}}\right\}\mathbf{h}_{\omega} = \mathbf{h}_{\omega}^H \mathbf{R}_x \mathbf{h}_{\omega}\,, \tag{31}$$

subject to the constraint that its frequency response at the frequency of interest ω has unity gain

$$H(\omega) = \sum_{k=0}^{K-1} h_{\omega}(kT) e^{-j\omega kT} = \mathbf{E}_{\omega}^T \mathbf{h}_{\omega} = 1\,, \tag{32.1}$$

$$H(\omega) = \sum_{k=0}^{K-1} h_{\omega}^*(kT) e^{j\omega kT} = \mathbf{h}_{\omega}^H \mathbf{E}_{\omega}^* = 1\,, \tag{32.2}$$

where $\mathcal{E}\{.\}$ denotes the expectation operator and the matrix $\mathbf{E}_{\omega}$ (Kx1) has elements $e^{-j\omega kT}$. The constraints (32.1) and (32.2) must be satisfied by the filter (30) and by the Hermitian transpose filter $y_{\omega}^H(nT) = \mathbf{h}_{\omega}^H \tilde{\mathbf{x}}^H$, correspondingly. The matrix $\mathbf{R}_x = \mathcal{E}\left\{\tilde{\mathbf{x}}^H \tilde{\mathbf{x}}\right\}$ (KxK) is the sample autocorrelation matrix and it can be composed from the values of the signal autocorrelation function

$$\mathbf{R}_x = \begin{bmatrix} r_{0,0}(0) & r_{0,1}(-T) & r_{0,2}(-2T) & ... & r_{0,K-1}\big(-(K-1)T\big) \\ r_{1,0}(T) & r_{1,1}(0) & r_{1,2}(-T) & ... & r_{1,K-1}\big(-(K-2)T\big) \\ r_{2,0}(2T) & r_{2,1}(T) & r_{2,2}(0) & ... & r_{2,K-1}\big(-(K-3)T\big) \\ ... & ... & ... & ... & ... \\ r_{K-1,0}\big((K-1)T\big) & r_{K-1,1}\big((K-2)T\big) & r_{K-1,2}\big((K-3)T\big) & ... & r_{K-1,K-1}(0) \end{bmatrix}, \tag{33}$$

for example, so called biased estimate calculated by

$$r_{xx}(lT) = \frac{1}{K}\sum_{k=0}^{K-l-1} x\big((k+l)T\big) x^*(kT), \quad l = 0,1,2,..., K-1\,. \tag{34}$$

Mathematically, the Capon filter coefficients can be obtained by minimizing the variance (31) under the constrains given by (32.1) and (32.2)

$$J = \mathbf{h}_{\omega}^H \mathbf{R}_x \mathbf{h}_{\omega} - \mu(\mathbf{E}_{\omega}^T \mathbf{h}_{\omega} - 1) - \lambda(\mathbf{h}_{\omega}^H \mathbf{E}_{\omega}^* - 1) = \min\,, \tag{35}$$

where μ, λ are Lagrange multipliers. The conditions $\dfrac{\partial J}{\partial \mathbf{h}_{\omega}} = 0$ and $\dfrac{\partial J}{\partial \mathbf{h}_{\omega}^H} = 0$ have to be fulfilled to determine the minimum of (35). Both requirements lead to the same solution

$$\mathbf{h}_{\omega} = \frac{\mathbf{R}_x^{-1}\mathbf{E}_{\omega}^*}{\mathbf{E}_{\omega}^T \mathbf{R}_x^{-1} \mathbf{E}_{\omega}^*}\,. \tag{36}$$

and, traditionally, the Capon power spectrum is computed as

$$P_{Capon}(\omega) = \mathbf{h}_{\omega}^H \mathbf{R}_x \mathbf{h}_{\omega} = \frac{1}{\mathbf{E}_{\omega}^T \mathbf{R}_x^{-1} \mathbf{E}_{\omega}^*}\,. \tag{37}$$

In order to obtain an iterative EDFT algorithm from the original Capon filter approach, the sample autocorrelation matrix $\mathbf{R}_x$ (33) has to be substituted by $\mathbf{R}^T = \frac{1}{N}\mathbf{E}^*\mathbf{W}\mathbf{E}^T$. The matrix $\mathbf{R}^T$ (KxK) can also be obtained as a transpose of the EDFT matrix **R** defined by (25) or (26). Consequently, the number of filters N must be greater than or equal to the length of the input sequence, $N \geq K$, which is a prerequisite for solving the system of linear equations (22). The elements of quadratic diagonal matrix **W** (NxN) represent an estimate of power at time moment nT=0, determined from one sample at the output of each Capon filter

$$|y_\omega(0)|^2 = |\tilde{\mathbf{x}}\mathbf{h}_\omega|^2 = \left|\frac{\tilde{\mathbf{x}}\left(\mathbf{R}^T\right)^{-1}\mathbf{E}_\omega^*}{\mathbf{E}_\omega^T\left(\mathbf{R}^T\right)^{-1}\mathbf{E}_\omega^*}\right|^2, \tag{38}$$

where the filter input sequence $\tilde{\mathbf{x}}$ (30) is related to the EDFT input sequence **x** as $\tilde{x}(kT) = x\left((K+k-1)T\right)$ or $\tilde{x}(t_k) = x(t_{K+k-1})$, k=0,-1,-2,..,-(K-1), for uniformly or nonuniformly sampled sequence cases, respectively.
Finally, an iterative algorithm with the initial condition for $\mathbf{W}^{(1)} = \mathbf{I}$ can be formed as follows

$$\mathbf{R}^T = \frac{1}{N}\mathbf{E}^*\mathbf{W}^{(i)}\mathbf{E}^T, \tag{39.1}$$

$$\mathbf{S}_{Capon}^{(i)} = \frac{\tilde{\mathbf{x}}\left(\mathbf{R}^T\right)^{-1}\mathbf{E}^*}{diag(\mathbf{E}^T\left(\mathbf{R}^T\right)^{-1}\mathbf{E}^*)}, \tag{39.2}$$

$$\mathbf{W}^{(i+1)} = diag(|\mathbf{S}^{(i)}|^2), \tag{39.3}$$

with the iteration number i=1,2,3,…I. Computer simulations for the iterative algorithm (39) shows that the estimate of the power spectrum $|\mathbf{S}_{Capon}|^2$ coincides with the results of the EDFT illustrated in Figure 1, while the phase spectrum, definitely, is different. In addition, an iterative algorithm derived on the basis of the Capon filter approach can not reveal all the EDFT capacity such as the ability to estimate the DFT (27.2) and restore the signal (29).

The Generalized Least Squares approach in the signal analysis is based on the following data model

$$\mathbf{x}^T = \mathbf{E}_\omega^* S_{GLS}(\omega) + \mathbf{e}_Q, \tag{40}$$

with $\mathbf{e}_Q$ denoting the noise and interference (signals at frequency grid points other than ω) term, and $\mathbf{E}_\omega^* S_{GLS}(\omega)$ representing the signal term on the frequency of interest with unknown complex amplitude $S_{GLS}(\omega)$. The GLS minimizes

$$[\mathbf{x}^T - \mathbf{E}_\omega^* S_{\mathrm{GLS}}(\omega)]^H \mathbf{Q}^{-1}[\mathbf{x}^T - \mathbf{E}_\omega^* S_{GLS}(\omega)], \tag{41}$$

which is solved by

$$S_{GLS}(\omega) = \frac{\mathbf{E}_\omega^T\mathbf{Q}^{-1}\mathbf{x}^T}{\mathbf{E}_\omega^T\mathbf{Q}^{-1}\mathbf{E}_\omega^*}, \tag{42}$$

where **Q** (KxK) is the covariance matrix of the data model term $\mathbf{e}_Q$. There are two special cases of GLS called Weighted Least Squares (WLS) and ordinary Least Squares (LS). WLS occurs when all the off-diagonal entries of **Q** are 0, while LS solution is obtained from the GLS under assumption that $\mathbf{e}_Q$ in (40) is a white noise, hence **Q**=**I**. The problem of GLS estimator is that, in general, the covariance matrix **Q** is not known, and must be estimated from the data along with the $S_{GLS}(\omega)$. The initial estimate (the 1st iteration) could be equal to LS solution, it is (42) with **Q**=**I**. Next, to ensure that the GLS solution works in an iterative way as EDFT do, covariance matrix **Q** should be replaced by $\mathbf{R}^T = \frac{1}{N}\mathbf{E}^*\mathbf{W}\mathbf{E}^T$. As a result, GLS solution (42) coincides with the EDFT formula (23.3)

$$S_{GLS}(\omega) = \frac{\mathbf{E}_\omega^T \left(\mathbf{R}^T\right)^{-1} \mathbf{x}^T}{\mathbf{E}_\omega^T \left(\mathbf{R}^T\right)^{-1} \mathbf{E}_\omega^*} = \frac{\mathbf{x}\mathbf{R}^{-1}\mathbf{E}_\omega}{\mathbf{E}_\omega^H \mathbf{R}^{-1}\mathbf{E}_\omega} = S_\alpha(\omega) \tag{43}$$

and, as shown in the previous sections can be successfully used to update the matrix **W** and for calculation of the amplitude spectrum iteratively. Although such a substitution would be easy done, it is not supported by GLS data model (40), from where the matrix **Q** represents the data model term $\mathbf{e}_Q$ only and the signal term $\mathbf{E}_\omega^* S_{GLS}(\omega)$ must be excluded from it, whereas the matrix $\mathbf{R}^T$ is calculated for the entire signal $\mathbf{x}^T$, including $\mathbf{e}_Q$ and $\mathbf{E}_\omega^* S_{GLS}(\omega)$. The conclusion reached is that making the derivation of the iterative EDFT algorithm possible, invalidates GLS minimization expression (41) which require separation of both data model terms.

The third method considered here is the High-Resolution DFT (HRDFT) available online at: http://www.sal.ufl.edu/eel6935/2008/00651165_SacchiUlrychWalker1998.pdf.
The authors presented an iterative nonparametric approach of spectral estimation, which minimizes the cost function deduced from Bayes' theorem and, like as Extended DFT makes it possible to obtain high-resolution Fourier spectrum. The HRDFT algorithm can be expressed by the following iterative procedure:

$$\mathbf{R} = \frac{1}{N}\mathbf{E}\mathbf{W}^{(i)}\mathbf{E}^H, \tag{44.1}$$

$$\mathbf{F}_{HRDFT}^{(i)} = \mathbf{x}\mathbf{R}^{-1}\mathbf{E}\mathbf{W}^{(i)}, \tag{44.2}$$

$$\mathbf{W}^{(i+1)} = diag\left(\frac{1}{N}\left|\mathbf{F}_{HRDFT}^{(i)}\right|^2\right), \tag{44.3}$$

for iteration number i=1,2,3,…I and with the initial condition $\mathbf{W}^{(1)} = \mathbf{I}$. The IDFT (28) applied for any iteration output (44.2), return back the sequence **x** undistorted. The main difference between approaches is that the HRDFT algorithm lack of formula for estimate of amplitude spectrum (27.3). Instead, as input for the next iteration, it uses the Fourier spectrum estimated in previous iteration (44.3). Therefore, the results of the HRDFT differ from output of the EDFT significantly. The simulation results depicted in Figures 1 and 2 allow compare the effectiveness of both methods and show that EDFT able to evaluate not only the spectrum of sinusoids, but also the shape of continuous spectrum of other signal terms, while HRDFT is suitable for the estimation of line spectra only.

EDFT algorithm in MATLAB code

The program NEDFT.m in MATLAB code is created to demonstrate the EDFT algorithm capabilities, which are described in previous sections. The program can be run for nonuniformly or uniformly sampled signals and for arbitrary selected frequency set f_n (see NEDFT.m program help for details). From the calculations complexity viewpoint, it is reasonable to select the frequencies on the same grid as used by the Fast Fourier Transform (FFT) algorithm. The program EDFT.m is designed as a faster realization of the EDFT algorithm **[4]**. This program is applicable for uniformly sampled signals and for signals with missing samples or data segments (gaps) inside of the input sequence (see EDFT.m program help for more details). The first version of the EDFT (file GDFT.m) was submitted on 10/7/1997 as MATLAB 4.1 code. The renewed code version submitted on 8/5/2006 and available online http://www.mathworks.com/matlabcentral/fileexchange/11020-extended-dft. Note that programs have not been tested under latest MATLAB versions, and therefore have a lot of space for performance improvements.

Selected reference articles

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[2] V. Liepin'sh, A method for spectrum evaluation applicable to analysis of periodically and non-regularly digitized signals, ACCS, Vol. 27, No. 6, pp. 57-64, 1993.
[3] V. Liepin'sh, A spectral estimation method of nonuniformly sampled band-limited signals, ACCS, Vol. 28, No. 2, pp. 66-73, 1994.
[4] V. Liepin'sh, An algorithm for evaluation a discrete Fourier transform for incomplete data, ACCS, Vol. 30, No. 3, pp. 27-40, 1996.
[5] Vilnis Liepins, High-resolution spectral analysis by using basis function adaptation approach /*in Latvian*/, Doctoral Thesis for Scientific Degree of Dr. Sc. Comp., University of Latvia, 1997. Abstract available on http://www.opengrey.eu/item/display/10068/330816 .

Afterwords

Many thanks to all around the world who have expressed an interest in the Extended DFT, or tried to incorporate EDFT algorithms in their research works and applications. Good luck! Here are some successful stories:
http://www.edi.lv/media/uploads/UserFiles/dasp-web/dasp-papers/Greitans_sampta01.pdf
http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.123.4737&rep=rep1&type=pdf
http://ismir2007.ismir.net/proceedings/ISMIR2007_p399_barbedo.pdf
http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?tp=&arnumber=4477348&isnumber=4626339
http://sisl.seas.harvard.edu/files/pub/BarbedoLopesWolfe08.pdf
http://plaza.ufl.edu/haohe/papers/RIAA.pdf
http://www.meps10.pwr.wroc.pl/submission/data/papers/16.1.pdf
http://www.springerlink.com/content/kj8w474v677n6880/
http://iopscience.iop.org/1538-3881/139/2/783/aj_139_2_783.text.html
http://www.eurasip.org/Proceedings/Eusipco/Eusipco2010/Contents/papers/1569292079.pdf
http://www.eurasip.org/proceedings/Eusipco/Eusipco2010/Contents/papers/1569290281.pdf
http://www.eurasip.org/Proceedings/Eusipco/Eusipco2010/Contents/papers/1569291187.pdf
http://d.wanfangdata.com.cn/Periodical_dlxtjqzdhxb201003002.aspx
http://140.124.72.88/LAB/ICASSP2011/pdfs/0004272.pdf
http://140.124.72.88/LAB/ICASSP2011/pdfs/0004304.pdf
http://www.sersc.org/journals/IJCA/vol4_no1/4.pdf
In the above articles the iterative EDFT algorithm can be recognized by alternative names, such as signal dependent transform, iterative DFT, high-resolution spectral analysis technique or iterative adaptive approach. Some of authors also present their own derivations of the algorithm. Unfortunately, their research efforts are currently limited to finding the right formula for calculating the amplitude spectrum **S** (38, 43) followed by an iterative approach to update it. So far the EDFT output **F** (27) is completely absent in the freely available online works of these authors. However, both outputs, when used together and properly applied, could be a very effective tool for the analysis of uniformly and nonuniformly sampled signals. As an example, the IDFT (29) applied to **F** allows to reconstruct, re-sample and extrapolate the input sequence **x**. The $|\mathbf{F}|^2$ divided by the length of the DFT provides a high-resolution estimate of the power spectral density function, while the output $|\mathbf{S}|^2$ estimates the power spectrum and could be used for calculation of the signal autocorrelation function. The ratio of **F**/**S** divided by the length of the sequence **x** shows how the frequency resolution of the EDFT changes in respect to the classical DFT (see attached MATLAB programs DEMOEDFT.m and DEMONEDFT.m).